Let us define logical operators $Z(c_1)$ and $X(\bar{c}_1)$

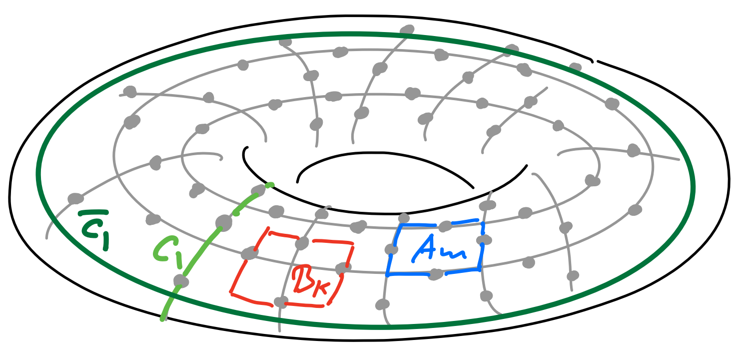
$\uparrow$ dual 1-chain

$\longrightarrow$ a) have to commute with all stabilizer generators

b) and be independent of them

a) $\longrightarrow$ $\partial c_1 = \partial \bar{c}_1 = 0$ (as $c_1 \cdot \delta v_k = \partial c_1 \cdot v_k = 0$ and $\bar{c}_1 \cdot \partial f_m = \partial \bar{c}_1 \cdot \bar{v}_m = 0$)

b) $c_1$ and $\bar{c}_1$ are non-trivial cycles

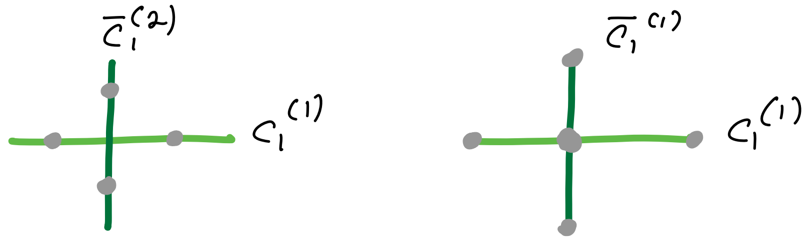



$\longrightarrow$ define two pairs of logical Pauli operators

$$\{L_z^{(1)} = Z(c_1^{(1)}),\ L_x^{(1)} = X(\bar{c}_1^{(1)})\}$$

$$\{L_z^{(2)} = Z(c_1^{(2)}),\ L_x^{(2)} = X(\bar{c}_1^{(2)})\}$$

Then




implies $$L_Z^{(i)} L_X^{(j)} = (-1)^{\delta_{ij}} L_X^{(j)} L_Z^{(i)}$$

equivalent to Pauli operators for two qubits.

Logical Pauli basis states are defined as follows:

$$L_Z^{(i)} |\psi_Z(s_1, s_2)\rangle = (-1)^{s_i} |\psi_Z(s_1, s_2)\rangle$$

$$L_X^{(i)} |\psi_X(s_1, s_2)\rangle = (-1)^{s_i} |\psi_X(s_1, s_2)\rangle$$

$\rightarrow$ number of stabilizer generators of the $n \times n$ square lattice on the torus is given by

$$|F| + |\bar{F}| - 2 = |F| + |V| - 2 = 2n^2 - 2$$

where the $-2$ comes from the fact that $\prod_{f_m \in F} A_m = I$ and $\prod_{v_k \in V} B_k = I$

(two non-independent operators)

$\#$ qubits $= |E| = 2n^2$

$\longrightarrow 2^{|E| - (|F| + |V| - 2)} = 2^2$ - dimensional stabilizer subspace

The above properties hold for general tilings $G = (V, E, F)$

$\rightarrow$ can define surface code for triangular, hexagonal etc. lattices

have the constraint:

$$|F| + |V| - |E| = 2 - 2g = -\chi_S$$

$\longrightarrow$ dimension stabilizer subspace

$= 2^{|E| - (|F| + |V| - 2)} = 2^{2g}$ ($2g$ qubits)

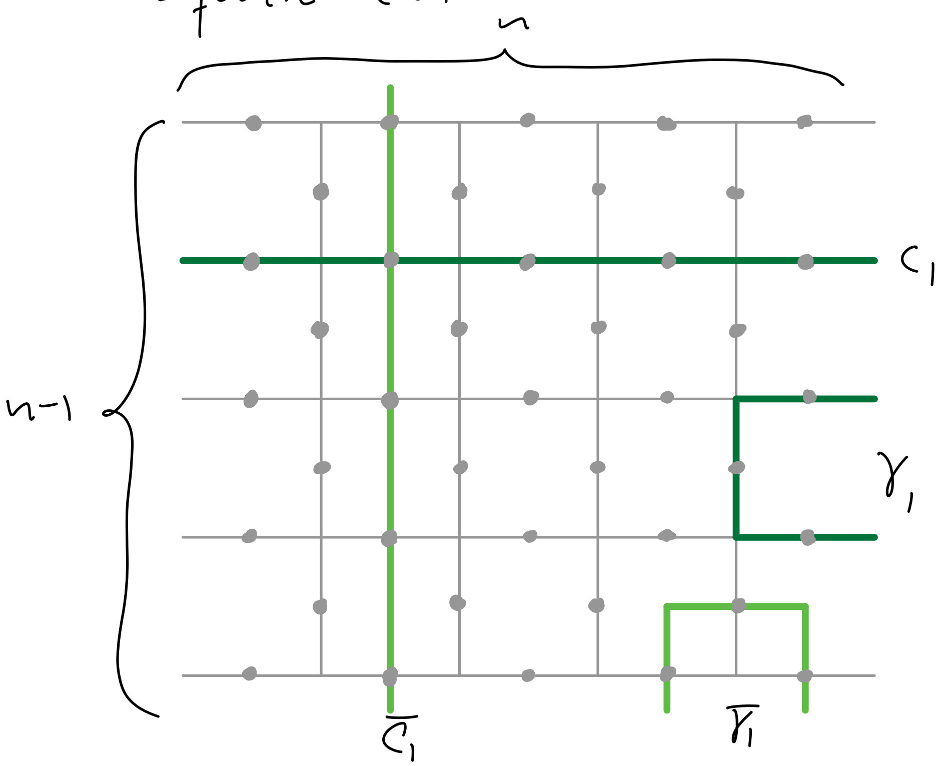
Schrödinger picture:

$$|\psi_Z(s_1, s_2)\rangle = Z(c_1^{(1)})^{s_1} Z(c_1^{(2)})^{s_2} \left( \prod_{v_k \in V} \frac{I + B_k}{2} \right) |0\rangle^{\otimes 2n^2}$$

$$|\psi_X(s_1, s_2)\rangle = X(\bar{c}^{(1)})^{s_1} X(\bar{c}_1^{(2)})^{s_2} \left( \prod_{f_m \in F} \frac{I + A_m}{2} \right) |+\rangle^{\otimes 2n^2}$$

## 2) Planar Surface Code

define surface code on planar $n \times (n-1)$ square lattice




We use relative homology to define logical operators: two chains $c_i$ and $c_i'$ are said to be (relative) homologically equivalent iff $c_i' = c_i + \partial c_{i+1} + \gamma_i$, $\gamma_i \in T_i \subset C_i$

$Z(\gamma_1)$ and $X(\bar{\gamma}_1)$ are stabilizer operators

$\rightarrow$ action of homological operators equivalent

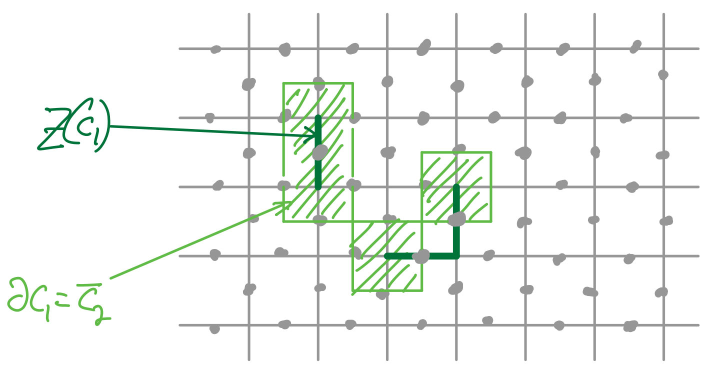
# § 3.4 Topological Quantum Error Correction

back to surface code on torus

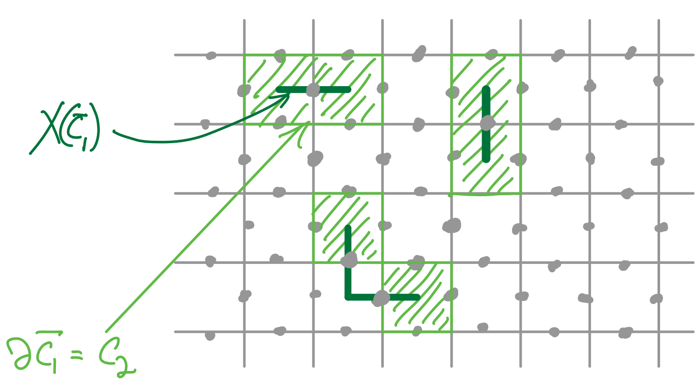
→ $2^{2n^2-2}$ orthogonal subspaces
(syndrome subspaces, eigenspaces of stabilizer generators)

→ can be utilized to identify location of errors. Suppose the X and Z errors $X(\bar{c}_1^{\,e})$ and $Z(c_1^{\,e})$ occur

→ $X(\bar{c}_1^{\,e})$ and $Z(c_1^{\,e})$ anticommute with $A_m$ and $B_k$ on the face $f_m \in \partial\bar{c}_1^{\,e}$ and vertex $v_k \in \partial c_1^{\,e}$




→ error syndrome specified by $\partial c_1^{\,e} = c_0^{\,s}$




→ error syndrome specified by $\partial \bar{c}_1^{\,e} = c_2^{\,s}$

More precisely, the error switches the eigenvalues of stabilizer generators $A_m$ and $B_k$ to $(-1)^{z_m^s}$ and $(-1)^{z_k^s}$ where

$$c_2^s = \sum_m z_m^s f_m \,, \quad c_0^s = \sum_k z_k^s v_k$$

Error correction is the task of finding recovery 1-chains $\bar{c}_1^r$ and $c_1^r$ such that

$$\partial(\bar{c}_1^e + \bar{c}_1^r) = 0 \,, \quad \partial(c_1^e + c_1^r) = 0$$

$\longrightarrow$ state is returned into code space by applying $X(\bar{c}_1^r)$ and $Z(c_1^r)$

Suppose each $Z$ error occurs with independent and identical probability $p$.

$\longrightarrow$ conditional probability of error $Z(c_1)$ with $c_1 = \sum_e z_e e_e$ :

$$P(c_1 | c_0^s) = \underset{\substack{\uparrow \\ \text{normalization}}}{\mathcal{N}} \prod_e \left(\frac{p}{1-p}\right)^{z_e} \Bigg|_{\partial c_1 = c_0^s}$$

→ find a recovery chain by maximizing posterior probability

$$c_1^r = \arg\max_{c_1} P(c_1 | c_0^s) = \arg\min\left(\sum_e z_e\right)\Big|_{\partial c_1 = c_0^s}$$

"minimum distance decoding"

If $c_1^r + c_1^e$ is trivial cycle, error correction succeeds

If $c_1^r + c_1^e$ is non-trivial cycle, we get logical error $Z(c_1^r + c_1^e) \sim L_Z$